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# Recursive Multibody Dynamics and Discrete-Time Optimal Control

G. M. T. D'Eleuterio and C. J. Damaren
Institute for Aerospace Studies
University of Toronto
Downsview, Ontario, Canada M3H 5T6

**Abstract**

A recursive algorithm is developed for the solution of the simulation dynamics problem for a chain of rigid bodies. Arbitrary joint constraints are permitted, that is, joints may allow translational and/or rotational degrees of freedom. The recursive procedure is shown to be identical to that encountered in a discrete-time optimal control problem. For each relevant quantity in the multibody dynamics problem, there exists an analog in the context of optimal control. The performance index that is minimized in the control problem is identified as Gibbs' function for the chain of bodies.

## 1 Introduction

The need to predict the motion of robotic systems in terrestial and and space applications has focused attention on the area of multibody dynamics. In this paper, we treat the simulation dynamics of a chain of rigid bodies. Given the external force distribution and control influences acting on the chain, we show how its subsequent motion, namely, the joint accelerations, can be determined using a recursive procedure. The equations of motion and kinematical constraints constitute a two-point boundary value problem. The key to its solution is the elimination of the constraint forces which exist at each joint. Our method in this regard is a generalization of that used by FEATHERSTONE [1983] for single degree of freedom joints, although FEATHERSTONE [1987] has explained how the extension to general constraints can be effected.

Recently, RODRIGUEZ [1987] has pointed out the similarity between the equations describing a chain of hinged bodies and those that arise in discrete-time optimal estimation and smoothing problems. His approach has utilized the correspondence with optimal filtering (the Kalman filter) and smoothing (the Bryson-Frazier smoother). Here, we show that the equations are identical in form to an optimal control problem. In fact, there is a one-to-one correspondence between the elements of the multibody dynamics problem and the control problem. The feedback solution for the control in terms of the state is precisely that which yields the joint accelerations in terms of the body accelerations. The analogy is further uncovered by identifying the performance index (written in terms of the chain dynamics) as GIBBS' [1879] function.

The major benefit of a recursive solution of the simulation problem is its computational efficiency. One avoids dealing with the system of equations describing the system in its entirety. This would involve Gaussian elimination of the global mass matrix at each time step. The computational consequences of this can be quite substantial since the number of calculations involved in a recursive solution grows linearly with the number of bodies whereas the Gaussian elimination obeys a cubic relationship.

## 2 Equations of Motion

Let us consider a chain of contiguous bodies $\mathcal{B}_0$, $\mathcal{B}_1, \ldots, \mathcal{B}_N$ as shown in Figure 1. Interbody joints may permit arbitrary relative (rotational and/or translational) motion. Each joint therefore possesses at least

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one degree of freedom and at most six. For convenience, we shall assume interbody translations to be small; however, the extension to large translations can be incorporated into the present formulation. For additional details on the derivation of the equations of motion, the reader should consult SINCARSIN & HUGHES [1989].

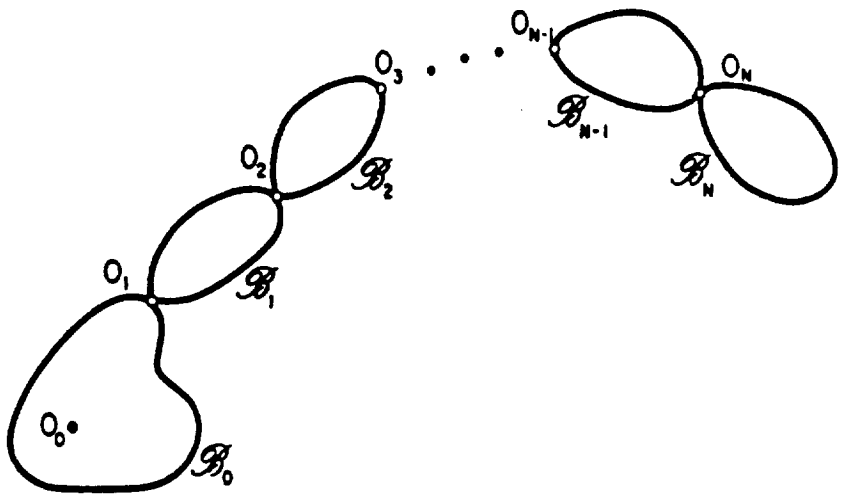


Figure 1: A Chain of Rigid Bodies

The motion of $\mathcal{B}_n$ is defined by the velocity $\mathbf{v}_n$ of $O_n$ and the angular velocity $\boldsymbol{\omega}_n$ of $\mathcal{B}_n$. (See Figure 2.) Both $\mathbf{v}_n$ and $\boldsymbol{\omega}_n$ are measured with respect to inertial space but are expressed in $\mathcal{F}_n$, a reference frame attached to $\mathcal{B}_n$. We shall define

$$v_n \triangleq \begin{bmatrix} \mathbf{v}_n \\ \boldsymbol{\omega}_n \end{bmatrix} \tag{1}$$

as the *generalized velocity* (*cf.* twist velocity) of $\mathcal{B}_n$ at $O_n$. We furthermore introduce the accompanying definition for a *generalized force* (*cf.* wrench) acting at $O_n$:

$$f_n^{n-1} \triangleq \begin{bmatrix} \mathbf{f}_n^{n-1} \\ \mathbf{g}_n^{n-1} \end{bmatrix} \tag{2}$$

where $\mathbf{f}_n^{n-1}$ and $\mathbf{g}_n^{n-1}$ are the reaction forces and torques on $\mathcal{B}_n$ due to $\mathcal{B}_{n-1}$ as expressed in $\mathcal{F}_n$.

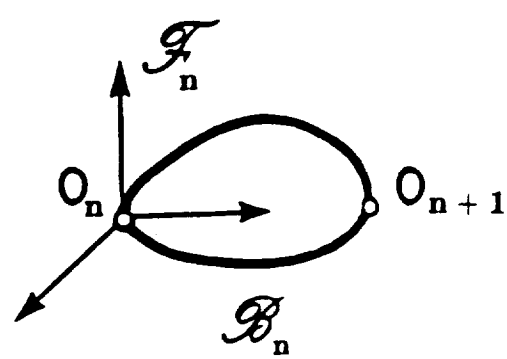


Figure 2: Reference Frame

The resulting equation of motion for $\mathcal{B}_n$ can be written as

$$\mathcal{M}_n \dot{v}_n = f_{nT} + f_{nI} \tag{3}$$

where

$$\mathcal{M}_n \triangleq \begin{bmatrix} m_n \mathbf{1} & -\mathbf{c}_n^\times \\ \mathbf{c}_n^\times & \mathbf{J}_n \end{bmatrix}$$

is the (constant) mass matrix corresponding to $\mathcal{B}_n$, that is, $m_n$, $\mathbf{c}_n$ and $\mathbf{J}_n$ are the zeroeth (mass), first and second moments of inertia (about $O_n$) of $\mathcal{B}_n$. Also, $\boldsymbol{f}_{nT}$ is the total external (generalized) force acting on $\mathcal{B}_n$, including interbody forces, and $\boldsymbol{f}_{nI}$, which accounts for the nonlinear inertial terms, can be neatly written as

$$\boldsymbol{f}_{nI} = (\boldsymbol{v}_n^\times)^T \mathcal{M}_n \boldsymbol{v}_n \tag{4}$$

where

$$\boldsymbol{v}_n^\times \triangleq \begin{bmatrix} \boldsymbol{\omega}_n^\times & \mathbf{v}_n^\times \\ \cdot & \boldsymbol{\omega}_n^\times \end{bmatrix}$$

and $(\cdot)^\times$ operating on a Cartesian $(3\times 1)$ column matrix, such as $\mathbf{v}_n$, $\boldsymbol{\omega}_n$ or $\mathbf{c}_n$, is the matrix equivalent of the vector cross product. In a rate-linear model, one would set $\boldsymbol{f}_{nI} \equiv 0$.

## Interbody Constraints

The set of equations (3) does not yet describe a chain of bodies since it does not take into consideration the interbody constraints imposed by the joints. To do so, we begin by observing that

$$\boldsymbol{v}_n = \mathcal{T}_{n,n-1}\boldsymbol{v}_{n-1} + \boldsymbol{v}_{n,\text{int}} \tag{5}$$

which introduces the *relative* interbody generalized velocity $\boldsymbol{v}_{n,\text{int}}$ of $\mathcal{B}_n$ with respect to $\mathcal{B}_{n-1}$. In addition,

$$\mathcal{T}_{n,n-1} = \begin{bmatrix} \mathbf{C}_{n,n-1} & -\mathbf{C}_{n,n-1}\mathbf{r}_{n-1}^{n\,\times} \\ \cdot & \mathbf{C}_{n,n-1} \end{bmatrix}$$

is the *generalized tranformation matrix* between $\mathcal{B}_{n-1}$ and $\mathcal{B}_n$; $\mathbf{C}_{n,n-1}$ is the rotation matrix from $\mathcal{F}_{n-1}$ to $\mathcal{F}_n$ and $\mathbf{r}_{n-1}^n$ is the position of $O_n$ with respect to $O_{n-1}$. The geometric constraints imposed by the joints can thus be expressed formally as

$$\boldsymbol{v}_{n,\text{int}} = \mathcal{P}_n \boldsymbol{v}_{n\gamma} \tag{6}$$

where $\mathcal{P}_n$ is a projection matrix and $\boldsymbol{v}_{n\gamma}$ is the column of free joint (rate) variables. The absolute velocities $\boldsymbol{v}_n$ can be obtained *recursively* from $\boldsymbol{v}_{n-1}$ and $\boldsymbol{v}_{n\gamma}$.

We also note that

$$\boldsymbol{f}_{nT} = \mathcal{T}_{n+1,n}^T \boldsymbol{f}_{n+1}^n - \boldsymbol{f}_n^{n-1} + \boldsymbol{f}_{n,\text{ext}} \tag{7}$$

where $\boldsymbol{f}_{n,\text{ext}}$ is due to solely to external influences. Furthermore, the generalized interbody forces $\boldsymbol{f}_n^{n-1}$ can be expressed as a sum of control forces $\boldsymbol{f}_{n,\text{c}}$ and constraint forces $\boldsymbol{f}_{n,\square}$, *i.e.*,

$$\boldsymbol{f}_n^{n-1} = -\mathcal{P}_n \boldsymbol{f}_{n,\text{c}} - \mathcal{Q}_n \boldsymbol{f}_{n,\square} \tag{8}$$

The projection matrix $\mathcal{Q}_n$ is the complement of $\mathcal{P}_n$.

## Projection Matrices

A few words are perhaps in order regarding the projection matrices. First, as a simple yet very important example, consider a joint with a single rotational degree of freedom about, say, the third axis of an appropriately chosen reference frame. The corresponding projection matrix $\mathcal{P}_n$ is

$$\mathcal{P}_n = [\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1\ ]^T$$

We may also add that $\boldsymbol{v}_{n\gamma} = \dot{\gamma}_3$, where $\gamma_3$ is the angle of rotation.

In general, $\mathcal{P}_n$ is not constant, as above, but rather is dependent on configuration. Contemplation of a universal joint will quickly reveal this fact. The columns of $\mathcal{P}_n$ are in general not orthonormal but

$$\mathcal{P}_n^T \mathcal{P}_n = \mathcal{I}_n \tag{9}$$

where is $\mathcal{I}_n$ is nonsingular. The complementary projection matrix $\mathcal{Q}_n$ satisfies

$$\mathcal{P}_n^T \mathcal{Q}_n = \mathbf{O} \tag{10}$$

Without loss in generality, the columns of $\mathcal{Q}_n$ can be taken as orthonormal.

### Kinematical Equations

The kinematical equations accompanying the dynamical equations (3) can be summarized in terms of $\mathcal{T}_{n,n-1}$:

$$\dot{\mathcal{T}}_{n,n-1} = -\boldsymbol{v}_{n,\text{int}}^{\times} \mathcal{T}_{n,n-1} \tag{11}$$

If we express $\boldsymbol{v}_{n,\text{int}}$ as

$$\boldsymbol{v}_{n,\text{int}} \triangleq \begin{bmatrix} \mathbf{v}_{n,\text{int}} \\ \boldsymbol{\omega}_{n,\text{int}} \end{bmatrix} \tag{12}$$

we can extract from (11),

$$\dot{\mathbf{C}}_{n,n-1} = -\boldsymbol{\omega}_{n,\text{int}}^{\times} \mathbf{C}_{n,n-1} \tag{13}$$

For physical reasons, Euler angles make for the most convenient and expedient representation of rotational joint degrees of freedom. Interbody translation is given by the integration of $\mathbf{v}_{n,\text{int}}$ and would be reflected in $\mathbf{r}_{n-1}^n$.

## 3 Rate-Linear Simulation Dynamics

The recursive method presented here is a generalization of Featherstone's method applicable to rigid multibody chains with arbitrary interbody constraints. The development, in fact, runs parallel to a similar generalization of Armstrong's recursive method [D'ELEUTERIO 1989]. The essential difference is that the former is based on an affine relationship of the total interbody force to the absolute (generalized) body acceleration while the latter relates explicitly only the interbody constraint force. The generalized Featherstone approach is particularly appealing because of its direct analogy to the discrete-time optimal problem. As shall be demonstrated, however, a simple equivalence exists between the two schemes.

Let us begin, for explanatory purposes, by considering the rate-linear model, that is, we shall set $\boldsymbol{f}_{nI} \equiv \mathbf{0}$ in (3) leaving

$$\mathcal{M}_n \dot{\boldsymbol{v}}_n = \boldsymbol{f}_{nT} \tag{14}$$

The extension to the nonlinear case (and, in fact, to elastic multibody trees) will be straightforward from here, although not totally without some algebraic effort.

### Recursion for $\boldsymbol{f}_n^{n-1}$

We conjecture that the interbody forces $\boldsymbol{f}_n^{n-1}$ can be written as

$$-\boldsymbol{f}_n^{n-1} = \boldsymbol{\Psi}_n \dot{\boldsymbol{v}}_n + \boldsymbol{\psi}_n \tag{15}$$

which is a generalization of Featherstone's hypothesis. Note that $\boldsymbol{\Psi}_n$ is, in effect, a mass matrix and $\boldsymbol{\psi}_n$ is a generalized force quantity. The recursive algorithm is based on this result and the fact that $\boldsymbol{\Psi}_n$ and $\boldsymbol{\psi}_n$ can be determined recursively from $\mathcal{B}_N$ to $\mathcal{B}_0$.

The proof of (15) is by induction:

**Step I.** For $\mathcal{B}_N$, (14) becomes

$$\mathcal{M}_N \dot{v}_N = -f_N^{N-1} + f_{N,\text{ext}} \tag{16}$$

wherein it has been observed that

$$f_{N+1}^N \equiv 0$$

since $\mathcal{B}_N$ is the (free) terminal body. It is immediately obvious that if we set

$$\Psi_N = \mathcal{M}_N, \quad \psi_N = -f_{N,\text{ext}} \tag{17}$$

(15) is satisfied for $n = N$.

**Step II.** We assume that

$$-f_{n+1}^n = \Psi_{n+1}\dot{v}_{n+1} + \psi_{n+1} \tag{18}$$

**Step III.** Given (18), we shall show that (14) follows. Substituting (7) into (14) yields

$$\mathcal{M}_n \dot{v}_n = \mathcal{T}_{n+1,n}^T f_{n+1}^n - f_n^{n-1} + f_{n,\text{ext}} \tag{19}$$

Now,

$$v_{n+1} = \mathcal{T}_{n+1,n} v_n + \mathcal{P}_{n+1} v_{n+1,\gamma} \tag{20}$$

and

$$\dot{v}_{n+1} = \mathcal{T}_{n+1,n} \dot{v}_n + \mathcal{P}_{n+1} \dot{v}_{n+1,\gamma} \tag{21}$$

(Note that the terms involving the time derivatives of $\mathcal{T}_{n+1,n}$ and $\mathcal{P}_{n+1}$ are omitted since they are nonlinear rate terms.) Substituting (21) and (8) in (19) and premultiplying by $\mathcal{P}_{n+1}^T$ gives

$$\mathcal{I}_{n+1} f_{n+1,c} = \Psi_{n+1,PP}\dot{v}_{n+1,\gamma} + \mathcal{P}_{n+1}^T \Psi_{n+1} \mathcal{T}_{n+1,n} \dot{v}_n + \psi_{n+1,P} \tag{22}$$

where, in general,

$$\Psi_{nPP} \triangleq \mathcal{P}_n^T \Psi_n \mathcal{P}_n, \quad \psi_{nP} \triangleq \mathcal{P}_n^T \psi_n$$

Solving for $\dot{v}_{n+1,\gamma}$ from (22), inserting back into (21) and using the result with (18) in (19) eventually leads to

$$\begin{aligned} -f_n^{n-1} &= \{\mathcal{M}_n + \mathcal{T}_{n+1,n}^T(\Psi_{n+1} - \Psi_{n+1}\mathcal{P}_{n+1}\Psi_{n+1,PP}^{-1}\mathcal{P}_{n+1}^T\Psi_{n+1})\mathcal{T}_{n+1,n}\}\dot{v}_n \\ &\quad + \{\mathcal{T}_{n+1,n}^T[\Psi_{n+1}\mathcal{P}_{n+1}\Psi_{n+1,PP}^{-1}(\mathcal{I}_{n+1}f_{n+1,c} - \psi_{n+1,P}) + \psi_{n+1}] - f_{n,\text{ext}}\} \end{aligned} \tag{23}$$

Hence, we can identify

$$\begin{aligned} \Psi_n &= \mathcal{M}_n + \mathcal{T}_{n+1,n}^T(\Psi_{n+1} - \Psi_{n+1}\mathcal{P}_{n+1}\Psi_{n+1,PP}^{-1}\mathcal{P}_{n+1}^T\Psi_{n+1})\mathcal{T}_{n+1,n} \\ \psi_n &= \mathcal{T}_{n+1,n}^T[\Psi_{n+1}\mathcal{P}_{n+1}\Psi_{n+1,PP}^{-1}(\mathcal{I}_{n+1}f_{n+1,c} - \psi_{n+1,P}) + \psi_{n+1}] - f_{n,\text{ext}} \end{aligned} \tag{24}$$

**Step IV.** By induction, then, (15) is proven. □

The matrix $\Psi_n$ has an attractive physical interpretation. It is the mass matrix (about $O_n$) of the part of the chain from $\mathcal{B}_n$ to $\mathcal{B}_N$ associated with the constrained degrees of freeedom. FEATHERSTONE [1983] would refer to $\Psi_n$ as the *articulated-body inertia*. It should also be pointed out that $\Psi_n$, which is positive-definite, and $\psi_n$ are configuration-dependent.

## Recursion for $\dot{v}_{n\gamma}$

By the inductive nature of the proof for (15), it has been shown that the matrices $\Psi_n$ and $\psi_n$ can be evaluated recursively inward, *i.e.*, from $\mathcal{B}_N$ to $\mathcal{B}_0$. Having done so, one can then perform outward recursion, from $\mathcal{B}_0$ to $\mathcal{B}_N$, to solve for $\dot{v}_{n\gamma}$. This is evident from (22).

Rewriting (22) for $\mathcal{B}_n$ instead of $\mathcal{B}_{n+1}$ and solving explicitly for $\dot{v}_{n\gamma}$ yields

$$\dot{v}_{n\gamma} = \Psi_{nPP}^{-1}(\mathcal{I}_n f_{n,c} - \mathcal{P}_n^T \Psi_n \mathcal{T}_{n,n-1}\dot{v}_{n-1} - \psi_{nP}) \tag{25}$$

Examining this result, we see that at $\mathcal{B}_n$ all the quantities on the right-hand side are known since $\dot{v}_{n-1}$ can be computed recursively from its inboard neighbor according to (21).

# 4 Nonlinear Simulation Dynamics

The extension to the nonlinear case can be had by simply persevering with the nonlinear rate terms in the preceding development. However, there is a much more palatable approach which is also not without significance in computational considerations.

Let [GOLLA 1988]

$$\dot{\boldsymbol{v}}_n = \boldsymbol{a}_n + \boldsymbol{a}_{n,\text{non}} \tag{26}$$

such that

$$\boldsymbol{a}_n = \boldsymbol{\mathcal{T}}_{n,n-1}\boldsymbol{a}_{n-1} + \boldsymbol{\mathcal{P}}_n\dot{\boldsymbol{v}}_{n\gamma} \tag{27}$$

Inserting (27) into (5) and differentiating reveals that we must have

$$\boldsymbol{a}_{n,\text{non}} = \boldsymbol{\mathcal{T}}_{n,n-1}\boldsymbol{a}_{n-1,\text{non}} + \dot{\boldsymbol{\mathcal{T}}}_{n,n-1}\boldsymbol{v}_{n-1} + \dot{\boldsymbol{\mathcal{P}}}_n\boldsymbol{v}_{n\gamma} \tag{28}$$

for (28) to hold. In essence, the acceleration quantities $\boldsymbol{a}_n$ account for the rate-linear effects and $\boldsymbol{a}_{n,\text{non}}$ for the nonlinear effects. Moreover, not only is $\boldsymbol{a}_n$ found recursively (outward) but $\boldsymbol{a}_{n,\text{non}}$ as well.

Upon substitution of (27) into the motion equation (3), we have

$$\boldsymbol{\mathcal{M}}_n\boldsymbol{a}_n = \boldsymbol{f}_{nT} + \boldsymbol{f}_{nI} + \boldsymbol{f}_{n,\text{non}} \tag{29}$$

where

$$\boldsymbol{f}_{n,\text{non}} \triangleq -\boldsymbol{\mathcal{M}}_n\boldsymbol{a}_{n,\text{non}}$$

In fact, we could write (30) as

$$\boldsymbol{\mathcal{M}}_n\boldsymbol{a}_n = \boldsymbol{\mathcal{T}}^T_{n+1,n}\boldsymbol{f}^n_{n+1} - \boldsymbol{f}^{n-1}_n + \boldsymbol{f}_{n,\text{net}} \tag{30}$$

where

$$\boldsymbol{f}_{n,\text{net}} \triangleq \boldsymbol{f}_{n,\text{ext}} + \boldsymbol{f}_{nI} + \boldsymbol{f}_{n,\text{non}}$$

Comparing (31) to (19), we learn that the nonlinear dynamics model is of the identical form as the rate-linear model with $\dot{\boldsymbol{v}}_n$ replaced by $\boldsymbol{a}_n$ and $\boldsymbol{f}_{n,\text{ext}}$ replaced by $\boldsymbol{f}_{n,\text{net}}$. We can therefore apply the results obtained above directly to the nonlinear case.

## Recursion for $\boldsymbol{f}^{n-1}_n$

In general, then, for rigid multibody chains

$$-\boldsymbol{f}^{n-1}_n = \boldsymbol{\Psi}_n\boldsymbol{a}_n + \boldsymbol{\wp}_n \tag{31}$$

Note that $\boldsymbol{\Psi}_n$ is the same as before; however,

$$\boldsymbol{\wp}_n = \boldsymbol{\mathcal{T}}^T_{n+1,n}[\boldsymbol{\Psi}_{n+1}\boldsymbol{\mathcal{P}}_{n+1}\boldsymbol{\Psi}^{-1}_{n+1,PP}(\boldsymbol{\mathcal{I}}_{n+1}\boldsymbol{f}_{n+1,\text{c}} - \boldsymbol{\wp}_{n+1,P})\boldsymbol{\wp}_{n+1}] - \boldsymbol{f}_{n,\text{net}} \tag{32}$$

with

$$\boldsymbol{\wp}_{nP} \triangleq \boldsymbol{\mathcal{P}}^T_n\boldsymbol{\wp}_n$$

and $\boldsymbol{\wp}_N = -\boldsymbol{f}_{N,\text{net}}$.

## Recursion for $\dot{\boldsymbol{v}}_{n\gamma}$

The recursive relation for $\dot{\boldsymbol{v}}_{n\gamma}$ can be expressed as

$$\dot{\boldsymbol{v}}_{n\gamma} = \boldsymbol{\Psi}^{-1}_{nPP}(\boldsymbol{\mathcal{I}}_n\boldsymbol{f}_{n,\text{c}} - \boldsymbol{\mathcal{P}}^T_n\boldsymbol{\Psi}_n\boldsymbol{\mathcal{T}}_{n,n-1}\boldsymbol{a}_{n-1} - \boldsymbol{\wp}_{nP}) \tag{33}$$

which reflects (26). It bears mentioning that the kinematical equations remain unchanged.

## Relationship to Armstrong's Work

Before proceeding onward, it is worth pointing out that

$$\mathbf{\Psi}_n - \mathbf{\Psi}_n \mathcal{P}_n \mathbf{\Psi}_{nPP}^{-1} \mathcal{P}_n^T \mathbf{\Psi}_n = \mathcal{Q}_n \mathbf{\Phi}_n \mathcal{Q}^T \tag{34}$$

where

$$\mathbf{\Phi}_n = \mathbf{\Psi}_{nQQ} - \mathbf{\Psi}_{nPQ}^T \mathbf{\Psi}_{nPP}^{-1} \mathbf{\Psi}_{nPQ}$$

and

$$\mathbf{\Psi}_{nPQ} \triangleq \mathcal{P}_n^T \mathbf{\Psi}_n \mathcal{Q}_n, \qquad \mathbf{\Psi}_{nQQ} \triangleq \mathcal{Q}_n^T \mathbf{\Psi}_n \mathcal{Q}_n$$

Showing (34) requires invoking the identity

$$\mathcal{P}_n \mathcal{I}_n^{-1} \mathcal{P}_n^T + \mathcal{Q}_n \mathcal{Q}_n^T = \mathbf{1}$$

By virtue of (34), we can rewrite the first of (24) as

$$\mathbf{\Psi}_n = \mathcal{M}_n + \mathcal{T}_{n+1,n}^T \mathcal{Q}_{n+1} \mathbf{\Phi}_{n+1} \mathcal{Q}_{n+1}^T \mathcal{T}_{n+1,n} \tag{35}$$

which is a more streamlined expression.

The significance of $\mathbf{\Phi}_n$, however, lies in the fact that

$$\boldsymbol{f}_{n,\square} = \mathbf{\Phi}_n \mathcal{Q}_n^T \boldsymbol{a}_n + \boldsymbol{\phi}_n \tag{36}$$

where

$$\boldsymbol{\phi}_n = \mathcal{Q}_n^T \boldsymbol{\wp}_n + \mathbf{\Psi}_{nPQ}^T \mathbf{\Psi}_{nPP}^{-1} (\mathcal{I}_n \boldsymbol{f}_{n,c} - \boldsymbol{\wp}_{nP})$$

This result is equivalent to Armstrong's method for rigid multibody chains with arbitrary joint constraints.

# 5 A Discrete-Time Optimal Control Problem

Diverting our attention from multibody dynamics momentarily, consider the following optimal control problem: minimize

$$\mathcal{J} = \sum_{k=0}^{N} \frac{1}{2} \mathbf{x}_k^T \mathbf{M}_k \mathbf{x}_k + \mathbf{x}_k^T \mathbf{h}_k - \mathbf{u}_{k-1}^T \mathbf{t}_{k-1} \tag{37}$$

subject to the linear state equation

$$\mathbf{x}_{k+1} = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k, \quad \mathbf{x}_{-1} = \mathbf{0} \tag{38}$$

Here, $\mathbf{M}_k$ is a sequence of positive-definite weighting matrices, and $\mathbf{h}_k$ and $\mathbf{t}_k$ are vector weighting sequences. Since $\mathbf{u}_N$ does not influence $\mathbf{x}_k$, $k \leq N$, we shall assume that $\mathbf{t}_N = \mathbf{0}$. This problem is slightly different than the standard "linear-quadratic" version that one typically encounters. The cost functional in the present case is linear in the control variable.

Minimizing $\mathcal{J}$ subject to the state equation is a straightforward optimization problem. Introducing the lagrange multiplier or adjoint variable $\boldsymbol{\lambda}_k$, we define the augmented performance index as follows:

$$\mathcal{J}' \triangleq \sum_{k=0}^{N} \frac{1}{2} \mathbf{x}_k^T \mathbf{M}_k \mathbf{x}_k + \mathbf{x}_k^T \mathbf{h}_k - \mathbf{u}_{k-1}^T \mathbf{t}_{k-1} + \boldsymbol{\lambda}_k^T (\mathbf{x}_k - \mathbf{A}_{k-1} \mathbf{x}_{k-1} - \mathbf{B}_{k-1} \mathbf{u}_{k-1}) \tag{39}$$

The necessary conditions for optimality,

$$\frac{\partial \mathcal{J}'}{\partial \boldsymbol{\lambda}_{k+1}} = \frac{\partial \mathcal{J}'}{\partial \mathbf{x}_k} = \frac{\partial \mathcal{J}'}{\partial \mathbf{u}_k} = \mathbf{0}$$

produce the two-point boundary value problem (TPBVP):

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}_k\mathbf{x}_k + \mathbf{B}_k\mathbf{u}_k\ , \quad \mathbf{x}_{-1} = \mathbf{0} && (40)\\ \boldsymbol{\lambda}_k &= \mathbf{A}_k^T\boldsymbol{\lambda}_{k+1} - \mathbf{M}_k\mathbf{x}_k - \mathbf{h}_k\ , \quad \boldsymbol{\lambda}_{N+1} = \mathbf{0} && (41)\\ & \mathbf{t}_k = -\mathbf{B}_k^T\boldsymbol{\lambda}_{k+1} && (42) \end{aligned}$$

We have taken $\boldsymbol{\lambda}_{N+1} = \mathbf{0}$, without loss in generality, since $\mathbf{t}_N = \mathbf{0}$. Hence, from (41), $\boldsymbol{\lambda}_N = -\mathbf{M}_N\mathbf{x}_N - \mathbf{h}_N$ which supplies the basis for the inhomogeneous Riccati transformation, sometimes called the sweep method:

$$\boldsymbol{\lambda}_k = -\mathbf{S}_k\mathbf{x}_k - \mathbf{r}_k \tag{43}$$

with $\mathbf{S}_N = \mathbf{M}_N$ and $\mathbf{r}_N = \mathbf{h}_N$. Substituting (43) into the equation for $\mathbf{t}_k$ (42) and replacing $\mathbf{x}_{k+1}$ with the right side of (40), produces the feedback law

$$\mathbf{u}_k = -\mathbf{K}_k\mathbf{x}_k + \mathbf{R}_k^{-1}[\mathbf{t}_k - \mathbf{B}_k^T\mathbf{r}_{k+1}] \tag{44}$$

where

$$\mathbf{R}_k \triangleq \mathbf{B}_k^T\mathbf{S}_{k+1}\mathbf{B}_k\ , \quad \mathbf{K}_k \triangleq \mathbf{R}_k^{-1}\mathbf{B}_k^T\mathbf{S}_{k+1}\mathbf{A}_k$$

The matrix $\mathbf{R}_k$ will be invertible if $\mathbf{B}_k$ is monic and $\mathbf{S}_{k+1}$ is positive-definite. Substituting the sweep solution (43) for $\boldsymbol{\lambda}_k$ and $\boldsymbol{\lambda}_{k+1}$ and using (40) for $\mathbf{x}_{k+1}$ and (44) for $\mathbf{u}_k$ gives

$$\begin{aligned} & [\mathbf{S}_k - \mathbf{A}_k^T(\mathbf{S}_{k+1} - \mathbf{S}_{k+1}\mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^T\mathbf{S}_{k+1})\mathbf{A}_k - \mathbf{M}_k]\mathbf{x}_k \\ = \ & -\mathbf{r}_k + (\mathbf{A}_k - \mathbf{B}_k\mathbf{K}_k)^T\mathbf{r}_{k+1} + \mathbf{K}_k^T\mathbf{t}_k + \mathbf{h}_k \end{aligned}$$

Since this must hold for general $\mathbf{x}_k$, the coefficient of $\mathbf{x}_k$ must vanish as well as the right hand side. Hence,

$$\mathbf{S}_k = \mathbf{A}_k^T(\mathbf{S}_{k+1} - \mathbf{S}_{k+1}\mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^T\mathbf{S}_{k+1})\mathbf{A}_k + \mathbf{M}_k \tag{45}$$

which is the discrete-time matrix Riccati equation and

$$\mathbf{r}_k = (\mathbf{A}_k - \mathbf{B}_k\mathbf{K}_k)^T\mathbf{r}_{k+1} + \mathbf{K}_k^T\mathbf{t}_k + \mathbf{h}_k \tag{46}$$

We now return to the question of the invertibility of $\mathbf{R}_k$. The definitions of $\mathbf{K}_k$ and $\mathbf{R}_k$ reveal that $(\mathbf{A}_k - \mathbf{B}_k\mathbf{K}_k)^T\mathbf{S}_{k+1}\mathbf{B}_k = \mathbf{0}$ which allows us to write the Riccati equation as

$$\mathbf{S}_k = (\mathbf{A}_k - \mathbf{B}_k\mathbf{K}_k)^T\mathbf{S}_{k+1}(\mathbf{A}_k - \mathbf{B}_k\mathbf{K}_k) + \mathbf{M}_k \tag{47}$$

Since $\mathbf{S}_N = \mathbf{M}_N$ is symmetic and positive-definite, $\mathbf{S}_k$ is symmetric and positive-definite (using backwards induction). Hence, $\mathbf{R}_k$ defined previously is positive-definite and is always invertible.

The optimal control policy can now be summarized as follows: one solves the Riccati equation (45) (or (47)) and the vector equation (46) backwards from $k = N$ to $k = 0$ using the boundary conditions $\mathbf{S}_N = \mathbf{M}_N$ and $\mathbf{r}_N = \mathbf{h}_N$. The optimal control can then be calculated using (44) while propagating the state forward using the state equation (40).

# 6 Relationship Between Optimal Control and Recursive Dynamics

The TPBVP generated by the previous optimal control problem (40-42) is identical in form to that of the multibody dynamics problem (30), (33), and

$$\mathcal{I}_n\boldsymbol{f}_{n,c} = -\mathcal{P}_n^T\boldsymbol{f}_n^{n-1} \tag{48}$$

which follows from premultiplying (8) by $\mathcal{P}_n^T$ while recognizing (9) and (10). Therefore, we make the following identifications:

$$\begin{array}{ccccccc} \mathbf{x}_k & \longleftrightarrow & \boldsymbol{a}_n & \qquad & \boldsymbol{\lambda}_k & \longleftrightarrow & \boldsymbol{f}_n^{n-1} \\ \mathbf{u}_k & \longleftrightarrow & \dot{\boldsymbol{v}}_{n+1,\gamma} & & \mathbf{h}_k & \longleftrightarrow & -\boldsymbol{f}_{n,\text{net}} \\ \mathbf{A}_k & \longleftrightarrow & \mathcal{T}_{n+1,n} & & \mathbf{M}_k & \longleftrightarrow & \mathcal{M}_n \\ \mathbf{B}_k & \longleftrightarrow & \mathcal{P}_{n+1} & & \mathbf{t}_k & \longleftrightarrow & \mathcal{I}_{n+1}\boldsymbol{f}_{n+1,c} \end{array}$$

Hence, the accelerations $\boldsymbol{a}_n$ are analogous to the states, the interbody forces $\boldsymbol{f}_n^{n-1}$ are analogous to the adjoint states, the joint accelerations $\dot{\boldsymbol{v}}_{n\gamma}$ play the role of the control inputs, and the projection matrices $\boldsymbol{\mathcal{P}}_{n+1}$ take the place of the input matrix $\mathbf{B}_k$. It can be shown that the interbody tranformation matrices $\boldsymbol{\mathcal{T}}_{n+1,n}$ possess the properties of the state transition matrix thus completing the analogy. Comparing the transformation (43) with the generalization of Featherstone's solution (32) allows us to identify

$$\mathbf{S}_k \longleftrightarrow \boldsymbol{\Psi}_n \ , \quad \mathbf{r}_k \longleftrightarrow \boldsymbol{\wp}_n \ , \quad \mathbf{R}_k \longleftrightarrow \boldsymbol{\Psi}_{n+1,PP}$$

We also emphasize that recursion in time ($k$) has been replaced by spatial recursion ($n$) at a given instant in time.

Using the above identifications, the performance index $\mathcal{J}$ can be written as

$$\mathcal{J} = \sum_{n=0}^{N} \frac{1}{2} \boldsymbol{a}_n^T \boldsymbol{\mathcal{M}}_n \boldsymbol{a}_n - \boldsymbol{f}_{n,\text{net}}^T \boldsymbol{a}_n - \boldsymbol{f}_{n,\text{c}}^T \boldsymbol{\mathcal{I}}_n \dot{\boldsymbol{v}}_{n\gamma}$$

Hence, in the multibody dynamics problem one can minimize $\mathcal{J}$ subject to the kinematical constraint equation (30) to arrive at the defining equations. Compare this with GIBBS' [1879] formulation of the dynamics of a system of $N$ particles with masses $m_n$, coordinates $x_n, y_n, z_n$, and subjected to forces $X_n, Y_n, Z_n$: minimize

$$\sum_{n=1}^{N} \frac{1}{2} m_n (\ddot{x}_n^2 + \ddot{y}_n^2 + \ddot{z}_n^2) - X_n \ddot{x}_n - Y_n \ddot{y}_n - Z_n \ddot{z}_n$$

subject to the kinematical constraints.

In the work of RODRIGUEZ [1987], he points out the similarity between the equations describing a chain of hinged bodies and the TPBVP that arises in discrete-time, optimal estimation and smoothing problems. In his formulation, the bodies in the chain are numbered inwardly (*i.e.*, the tip body is $\mathcal{B}_0$ and the root body is $\mathcal{B}_N$). Here, the numbering of the bodies is outward (the root body is $\mathcal{B}_0$ and the tip body is $\mathcal{B}_N$). With this convention, the equations are rendered *dual* to those of Rodriguez. As such, the corresponding discrete-time problem is not one of estimation and smoothing but one of control. It is interesting to note the dual relationships inherent in Rodriguez's work. The role of the state is played by the interbody forces and the adjoint states are the link accelerations, which are a juxtaposition of the results given above. The control torque at each joint plays the role of a measurement of the states whereas we have the joint accelerations acting as 'control inputs'.

# 7 Summary of the Recursive Algorithm

We now summarize the procedures for determining the motion of the chain of bodies. The control forces, $\boldsymbol{f}_{n,\text{c}}(t)$, and external force distribution, $\boldsymbol{f}_{n,\text{ext}}(t)$, are prescribed on the time interval of interest. Beginning with $t = 0$, we proceed as follows:

**Step 1.** At time $t$, the relative velocities $\boldsymbol{v}_{n\gamma}(t)$ and the rotation matrices $\mathbf{C}_{n,n-1}(t)$ are known.

**Step 2.** Outward recursion for the velocities $\boldsymbol{v}_n$ and determination of $\boldsymbol{f}_{n,\text{net}}$:

Do $n = 0$ to $N$;

Generate $\boldsymbol{\mathcal{T}}_{n,n-1}$ using $\mathbf{C}_{n,n-1}$.

$\boldsymbol{v}_{n,\text{int}} = \boldsymbol{\mathcal{P}}_n \boldsymbol{v}_{n\gamma}$.

$\dot{\boldsymbol{\mathcal{T}}}_{n,n-1} = -\boldsymbol{v}_{n,\text{int}}^{\times} \boldsymbol{\mathcal{T}}_{n,n-1}$.

$\boldsymbol{v}_n = \boldsymbol{\mathcal{T}}_{n,n-1} \boldsymbol{v}_{n-1} + \boldsymbol{v}_{n,\text{int}}$.

$\boldsymbol{a}_{n,\text{non}} = \boldsymbol{\mathcal{T}}_{n,n-1} \boldsymbol{a}_{n-1,\text{non}} + \dot{\boldsymbol{\mathcal{T}}}_{n,n-1} \boldsymbol{v}_{n-1} + \dot{\boldsymbol{\mathcal{P}}}_n \boldsymbol{v}_{n\gamma}$.

$\boldsymbol{f}_{nI} = (\boldsymbol{v}_n^{\times})^T \boldsymbol{\mathcal{M}}_n \boldsymbol{v}_n$, $\boldsymbol{f}_{n,\text{non}} = -\boldsymbol{\mathcal{M}}_n \boldsymbol{a}_{n,\text{non}}$.

$\boldsymbol{f}_{n,\text{net}} = \boldsymbol{f}_{n,\text{ext}} + \boldsymbol{f}_{nI} + \boldsymbol{f}_{n,\text{non}}$.

Next n.

**Step 3.** Inward Recursion for $\boldsymbol{\Psi}_n$ and $\boldsymbol{\wp}_n$:

Set $\boldsymbol{\Psi}_N = \boldsymbol{\mathcal{M}}_N$ and $\boldsymbol{\wp}_N = -\boldsymbol{f}_{N,\text{non}}$.

Do $n = N-1$ to 0;

$\boldsymbol{\Psi}_{n+1,PP} = \boldsymbol{\mathcal{P}}_{n+1}^T \boldsymbol{\Psi}_{n+1} \boldsymbol{\mathcal{P}}_{n+1}$, $\boldsymbol{\wp}_{n+1,P} = \boldsymbol{\mathcal{P}}_{n+1}^T \boldsymbol{\wp}_{n+1}$

$\boldsymbol{\mathcal{K}}_n = \boldsymbol{\Psi}_{n+1,PP}^{-1} \boldsymbol{\mathcal{P}}_{n+1} \boldsymbol{\Psi}_{n+1} \boldsymbol{\mathcal{T}}_{n+1,n}$, $\boldsymbol{\Gamma}_{n+1,n} = \boldsymbol{\mathcal{T}}_{n+1,n} - \boldsymbol{\mathcal{K}}_n \boldsymbol{\mathcal{P}}_{n+1}$.

$\boldsymbol{\Psi}_n = \boldsymbol{\Gamma}_{n+1,n}^T \boldsymbol{\Psi}_{n+1} \boldsymbol{\Gamma}_{n+1,n} + \boldsymbol{\mathcal{M}}_n$

$\boldsymbol{\wp}_n = \boldsymbol{\Gamma}_{n+1,n}^T \boldsymbol{\wp}_{n+1} + \boldsymbol{\mathcal{K}}_n^T \boldsymbol{f}_{n+1,\text{c}} - \boldsymbol{f}_{n,\text{net}}$

Next n.

If $\boldsymbol{\mathcal{P}}_0 \neq \mathbf{O}$, $\boldsymbol{\Psi}_{0PP} = \boldsymbol{\mathcal{P}}_0^T \boldsymbol{\Psi}_0 \boldsymbol{\mathcal{P}}_0$, $\boldsymbol{\wp}_{0P} = \boldsymbol{\mathcal{P}}_0^T \boldsymbol{\wp}_0$

**Step 4.** Outward Recursion for $\dot{\boldsymbol{v}}_{n\gamma}$:

If $\boldsymbol{\mathcal{P}}_0 = \mathbf{O}$ ($\mathcal{B}_0$ is constrained), then $\dot{\boldsymbol{v}}_{0\gamma} = \boldsymbol{a}_0 = 0$

Otherwise, $\dot{\boldsymbol{v}}_{0\gamma} = \boldsymbol{\Psi}_{0PP}^{-1}[\boldsymbol{f}_{0,\text{c}} - \boldsymbol{\psi}_{0P}]$, $\boldsymbol{a}_0 = \boldsymbol{\mathcal{P}}_0 \boldsymbol{v}_{0\gamma}$.

Do $n = 1$ to $N$;

$\dot{\boldsymbol{v}}_{n\gamma} = -\boldsymbol{\mathcal{K}}_{n-1} \boldsymbol{a}_{n-1} + \boldsymbol{\Psi}_{n,PP}^{-1}(\boldsymbol{f}_{n,\text{c}} - \boldsymbol{\wp}_{nP})$

$\dot{\mathbf{C}}_{n,n-1} = -\boldsymbol{\omega}_{n,\text{int}}^{\times} \mathbf{C}_{n,n-1}$

$\boldsymbol{a}_n = \boldsymbol{\mathcal{T}}_{n,n-1} \boldsymbol{a}_{n-1} + \boldsymbol{\mathcal{P}}_n \dot{\boldsymbol{v}}_{n\gamma}$

Next n.

**Step 5.** Estimate $\boldsymbol{v}_{n\gamma}(t+\Delta t)$, $\mathbf{C}_{n,n-1}(t+\Delta t)$ using some quadrature scheme.

Go back to Step 1 and replace $t$ with $t + \Delta t$.

This completes the summary of the recursive simulation procedure. Note that in a rate-linear simulation, one ignores the contributions of $\boldsymbol{f}_{nI}$ and $\boldsymbol{f}_{n,\text{non}}$ to $\boldsymbol{f}_{n,\text{net}}$ in Step 2. We have written the recursion for $\boldsymbol{\Psi}_n$ and $\boldsymbol{\wp}_n$, in Step 3, in terms of the quantities $\boldsymbol{\mathcal{K}}_n$ and $\boldsymbol{\Gamma}_{n+1,n}$ since this leads to the most compact and efficient expressions. The fourth step produces the joint acclerations $\dot{\boldsymbol{v}}_{n\gamma}$ which can be integrated in conjunction with the kinematical relationships for the rotation matrices to produce the joint orientations/positions and velocities.

# 8 Concluding Remarks

Given the forces on a chain of rigid bodies, we have shown that the accelerations of the bodies can be determined using the recursive procedures of discrete-time optimal control. The underlying analogy that makes this possible yields great insight into the structure of the multibody dynamics problem.

There are many extensions of the present results, a few of which we shall briefly mention here. The analysis presented was limited to topological chains of rigid bodies. It is easily extended to topological tree configurations. The problem of flexible multibody dynamics has been considered by D'ELEUTERIO [1989] who shows that the structure of the equations is unaltered by flexibility. Indeed there is a one-to-one correspondence between the rigid and flexible problems. With this duality in hand, one can readily extend the present analysis to the problem of elastic multibody chains. Such an extension has been performed by DAMAREN & D'ELEUTERIO [1989].

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